Problem 1 (10 Points): Compute the inverse Fourier transform of the following signal:

\[ X(\omega) \]

\[ \sin^2 \left( \frac{\omega - 20}{\pi} \right) \]

Consider \( X_1(\omega) \)

\[ 2 \sin^2 \left( \frac{\omega}{\pi} \right) \]

\[ X_1(t) = \left( 1 - \frac{2|t|}{4} \right) \text{rect}(t) \]

Then \( x(t) = X_1(t) \cos(20t) \)
Problem 2 (20 Points): Given the following circuit:

\[ x(t) = 2 + 2 \cos(t) + 2 \cos(40t + \pi/2) \]

1. a) Compute \( H(\omega) \), and plot \(|H(\omega)|\) and \( \angle H(\omega) \) vs \( \omega \).

2. b) Compute the response of the circuit to the input

\[ y(t) = \frac{R}{R + \frac{1}{j\omega C}} x(t) \]

3. c) What kind of filter does this circuit represent?

\[ H(\omega) = \frac{RC/j\omega}{R + \frac{1}{j\omega C}} = \frac{1/j\omega}{1/j\omega + 1} \]

\[ |H(\omega)| = \frac{1}{\sqrt{(1\omega)^2 + 1}} \quad \text{and} \quad \angle H(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega) \]

\[ b) \quad H(0) = 0, \quad H(1) = \frac{1}{1 + 1} = 0.995 \quad \text{e}^{1.471} \]

\[ H(40) = \frac{4}{4 + 1} = 0.797 \quad \text{e}^{0.245} \]

\[ y(t) = 1.99 \cos(t + 1.47) + 1.94 \cos(40t + 1.82) \]

\[ c) \text{ high pass} \]
Problem 3 (15 Points): A linear time invariant system has the following impulse response:

\[ h(t) = 2e^{-at}u(t) \]

Use convolution to find the response \( y(t) \) to the following input:

\[ x(t) = u(t) - u(t - 4) \]

Sketch \( y(t) \) for the case when \( a = 1 \).

\[ y(t) = \begin{cases} 
0 & t < 0 \\
\frac{2}{a}(1 - e^{-at}) & 0 \leq t < 4 \\
\frac{2}{a}(e^{-a(t-4)} - e^{-at}) & t \geq 4 
\end{cases} \]

if \( a = 1 \)

\[ y(t) = \begin{cases} 
0 & t < 0 \\
\frac{2}{1}(1 - e^{-t}) & 0 \leq t < 4 \\
\frac{2}{1}(e^{-(t-4)} - e^{-t}) & t \geq 4 
\end{cases} \]
Problem 4 (20 Points): Consider the system defined in Problem 3.

a) Compute the Frequency response \( H(\omega) \) and \( X(\omega) \). Use this to compute \( Y(\omega) \). Sketch the magnitude and phase of \( Y(\omega) \) vs \( \omega \) for the case when \( a = 1 \).

\[
h(t) = 2e^{-at} u(t) \quad \iff \quad H(\omega) = \frac{2}{j\omega + a}
\]

\[
x(t) = p_d(t - 2) \quad \iff \quad X(\omega) = \frac{4\sin\left(\frac{t\omega}{a}\right)}{2\omega} e^{-2j\omega} = \frac{4}{2\omega} \sin\omega e^{-2j\omega}
\]

b) Use the information gained from part a) along with the information gained from Problem 3 to sketch the responses \( y(t) \) for the cases \( a = 0.1 \) and \( a = 10 \). In both cases, sketch the corresponding \(|Y(\omega)|\)'s.

\[
|H(\omega)| (a = 1)
\]

\[
x(t)
\]

\[
|X(\omega)|
\]

\[
y(t)
\]

\[
|Y(\omega)|
\]

\[
\alpha = 1: \quad Y(\omega) = \frac{2}{j\omega + 1} \cdot \frac{4\sin\omega}{\omega} e^{-2j\omega}
\]

\[
\text{where} \quad |Y(\omega)| = \frac{|H(\omega)|}{|X(\omega)|}
\]

\[
\text{and} \quad Y(\omega) = |H(\omega)| + |X(\omega)|
If \( a = 0.1 \), \( |H(\omega)| \) is (very narrow) bandwidth is 0.1 rad/sec (low freq).

So, \( |Y(\omega)| \)

\[
y(t) = \begin{cases} 
20(1 - e^{-it}) & \text{if } 0 < t < 4 \\
20(e^{-1/(t-4)} - e^{-it}) & \text{if } 4 < t < 8 
\end{cases}
\]

If \( a = 10 \), \( |H(\omega)| \) is very broad

bandwidth is 10 rad/sec

So, \( Y(\omega) \)

\( y(t) \) is more late \( x(t) \)
since \( H(\omega) \) has such a large bandwidth.
Note, many people wrote

\[ u(t) - u(t-4) \mapsto \pi \delta(\omega) + \frac{i}{j\omega} - e^{i\omega}(\pi \delta(\omega) + \frac{i}{j\omega}) \]

\[ = \pi \delta(\omega)(1 - e^{-i\omega}) + \frac{i}{j\omega}(1 - e^{-i\omega}) \]

The first term is zero since

\[ 1 - e^{-i\omega} = 0 \text{ at } \omega = 0 \]

\[ = \frac{i}{j\omega} (1 - e^{-i\omega}) \]

\[ = \frac{i}{j\omega} e^{-i\omega} \left( e^{i\omega} - e^{-i\omega} \right) \]

\[ = \frac{2}{\omega} e^{-i\omega} \sin(\omega) \]

\[ = 4 e^{-i\omega} \operatorname{sinc}(\omega) \]

Which is found easier by realizing that

\[ u(t) - u(t-4) = p_4(t-2) \mapsto 4 \operatorname{sinc}(\frac{2\omega}{\pi}) e^{-2i\omega} \]
Problem 5 (10 Points): Match the time responses with the corresponding frequency responses.

1. d  2. e  3. a  4. b  5. c
Problem 6 (short answers, 25 Points total):

a) When do we use Fourier series vs Fourier transforms?

Both are used to get the frequency content of a signal. Fourier series are for periodic signals while Fourier transforms are for aperiodic signals.

b) What is aliasing? Given the signal below, what is the minimum sampling rate (that is, the Nyquist rate) to avoid aliasing?

Aliasing is a distortion caused by sampling at too low of a rate. Nyquist rate is $W_N = 10 \text{ rad/sec}$.

c) The step response of a linear time-invariant system is given below.

$x(t) = u(t)$, $y(t) = (1 - e^{-2t})u(t)$

What is the impulse response? What is the response to $x(t) = 4u(t) - 4u(t-1)$?

$r(t) = \frac{du}{dt}$ so $h(t) = \frac{dy}{dt} = 2e^{-2t}u(t)$

Response to $x(t) = 4u(t) - 4u(t-1)$ is

$4\left(1 - e^{-2t}\right)u(t) - 4\left(1 - e^{-2(t-1)}\right)u(t-1)$

by linearity and time delay properties.
d) Give \( x(t) \).

\[
x(t) = \frac{1}{\pi} e^{10\pi t}
\]

\[ x(t) = 1 + \underbrace{2 + 2\cos(4t) + 3\cos(6t)}_{\text{called } X(\omega)}
\]

What is the fundamental frequency?

\[
X(t) = 2 + e^{14t} + e^{-14t} + 1.5 e^{74t} + 1.5 e^{-74t}
\]

\[
\omega_0 = 2 \text{ rad/} \text{s}
\]

f) Is the following system linear? time-invariant? causal? Justify your answer.

\[
\frac{dy}{dt} + \sin(t)y = 4x(t)
\]

Linear: ODE with coefficients independent of \( y \)

Time-varying: coef. depends on \( t \)

Causal: does not depend on future values of \( x(t) \)

f) Who will win the Gator Bowl? (I have a degree from Notre Dame and a degree from Georgia Tech, so I'd be happy with either of your answers!)

2 Go Tech!